Ambidexterity in chromatic homotopy theory

Gabrielle Li

UIUC Graduate Student Homotopy Theory Seminar, Fall 2023

10/27/2023

Outline

1. Main theorem
2. Tools
   1. Ambidexterity and semiadditivity
   2. Symmetric monoidal dimensions
   3. Additive p-derivation and α-operation
   4. Nil-conservativity and detection principles
3. Application to $\mathrm{Sp}_{T(n)}$
4. Generalization

Main Theorem: statement

Theorem (Carmelli-Schlank-Yanovski)

The category $Sp_{T(n)}$ of $T(n)$-local spectra is ∞-semiadditive.

Main theorem: series of reduction

We inductively prove that $Sp_{T(n)}$ is ∞-semiadditive. Assuming that $\mathrm{Sp}_{T(n)}$ is m-semiadditive, to show that it's $(m+1)$-semiadditive, we need to show that the natural transformation

$$\mathrm{Nm}_{B^{m+1}C_p} : \varinjlim B^{m+1}C_p \to \varprojlim B^{m+1}C_p$$

is an isomorphism. Consider a fiber sequence of spaces

$$A \to E \to B^{m+1}C_p$$

where A and E are both m-finite. It reduces to find an m-good A with to show that $|A|$ is invertible in $\pi_0 S^0_{T(n)}$.

Definition (m-good)

A space A is called m-good if it is connected, m-finite with $\pi_m(A) \neq 0$ and all homotopy groups of A are p-groups.

Main theorem: series of reduction

We have a symmetric monoidal functor

$$\widehat{E}_n : \mathrm{Sp}_{T(n)} \xrightarrow{L_{K(n)}} \mathrm{Sp}_{K(n)} \xrightarrow{F_{E_n}} \widehat{\mathrm{Mod}}_{E_n} := \mathrm{Mod}_{E_n}^{K(n)}$$

to the $K(n)$-local E_n-module spectra, which induces a map

$$f : \pi_0 S^0_{T(n)} \to \pi_0 E_n = \mathbb{Z}_p[[u_1, ...u_{n-1}]]$$

that detects invertibility, i. e. $|A|$ is invertible in $\pi_0 S^0_{T(n)}$ if and only if $f(|A|)$ is invertible in $\pi_0 E_n$. In fact, the image $f(\pi_0 S^0_{T(n)})$ is contained in $\mathbb{Z}_p$ and we can identify $f(|A|)$ in $\pi_0 E_n$ with $|A|$ in $\pi_0 S^0_{T(n)}$. This boils down to show that the p-adic valuation v_p of $|A|$ is non-zero. On $\mathbb{Z}_p$, we have

$$v_p(\tilde{\delta}(x)) = v_p(x) - 1.$$

for the Fermat quotient

$$\tilde{\delta}(x) = \frac{x - x^p}{p}.$$

Main theorem: series of reduction

Our goal is to construct an operation $\delta : \pi_0(E_n) \to \pi_0(E_n)$ that extends $\tilde{\delta}$ on $\mathbb{Z}_p$ and show that $|B^m C_p| \in \pi_0(E_n)$ is non-zero, as we can express

$$\delta(A) = |A'| - |A''|$$

where both A', A'' are m-good. If $|A|$ is not zero, then one of $|A'|, |A''|$ has a lower p-adic valuation.

Main theorem: structure of the proof

The authors developed a general machinery to prove that an ∞-category is ∞-semiadditive. In the case of $\mathrm{Sp}_{T(n)}$, we need to prove the following hypotheses:

1. The ∞-category of $T(n)$-local spectra $\mathrm{Sp}_{T(n)}$ is 1-semiadditive. (Kuhn, 2004)
2. The functor $\widehat{E}_n : \mathrm{Sp}_{T(n)} \to \widehat{\mathrm{Mod}}_{E_n}$ detects invertibility.
3. The symmetric monoidal dimensions of the spaces $B^k C_p$ in the ∞-category $\widehat{\mathrm{Mod}}_{E_n}$ is rational and non-zero.

The third claim characterizes invertibility in $\widehat{\mathrm{Mod}}_{E_n}$, which follows from the computation Ravenel and Wilson did on the Morava K-theory of the Eilenberg-McLane spaces.

Tools: Ambidexterity and semiadditivity

Definition (Normed and iso-normed)

Given $q : A \to B$ between spaces and an ∞-category $\mathcal{C}$, we have a restriction functor between the $\mathcal{C}$-valued local system:

$$q^* : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C}) \ .$$

Assuming that $\mathcal{C}$ admits q-(co)limits (i. e. (co)limits in the shape of the homotopy fiber $q^{-1}(b)$ for all $b \in B$), then q^* admits both a left adjoint $q_!$ and a right adjoint q_* (given by the left and right Kan extension, respectively). If there exists a map, which we call the norm map

$$Nm_q : q_! \to q_*$$

we call q normed. If Nm_q is an equivalence, we call q iso-normed.

Let $c_!$ denote the counit of the adjunction $q_! \dashv q^*$ and u_* denote the unit of the adjunction $q^* \dashv q_*$.

Definition (Base-change square)

Given an ∞-category $\mathcal{C}$ and a pullback diagram of spaces

$$\begin{array}{ccc} \tilde{A} & \xrightarrow{s_A} & A \\ {\scriptstyle \tilde{q}}\downarrow & & \downarrow{\scriptstyle q} \\ \tilde{B} & \xrightarrow[s_B]{} & B \end{array} .$$

The associated base-change square of $\mathcal{C}$-valued local systems is

$$\begin{array}{ccc} \mathrm{Fun}(B, \mathcal{C}) & \xrightarrow{s_B} & \mathrm{Fun}(\tilde{B}, \mathcal{C}) \\ {\scriptstyle q^*}\downarrow & & \downarrow{\scriptstyle \tilde{q}^*} \\ \mathrm{Fun}(A, \mathcal{C}) & \xrightarrow[s_A]{} & \mathrm{Fun}(\tilde{A}, \mathcal{C}) \end{array} .$$

It's called iso-normed if both q and $\tilde{q}$ is iso-normed.

Definition (Beck-Chevalley conditions)

Given the base-change as above, we can define the Beck-Chevalley transformation

$$\beta_! : \tilde{q}_! s_A \xrightarrow{u_i} \tilde{q}_! s_A q^* q_! \xrightarrow{\cong} \tilde{q}_! \tilde{q}^* s_B q_! \xrightarrow{\tilde{c}_!} s_B q_!$$

$$\beta_* : s_A q_* \xrightarrow{\tilde{u}_*} \tilde{q}_* \tilde{q}^* s_A q_* \xrightarrow{\cong} \tilde{q}_* s_B q^* q_* \xrightarrow{c_*} s_B q_!$$

If $\beta_!$ (resp. β_*) is an isomorphism, we say that the base-change square satisfies the $BC_!$ (resp. BC_*) condition.

Lemma (Criterion for $BC_!/BC_*$ condition)

In the setting above, if $\mathcal{C}$ admits all q-colimits (resp. q-limits), then the associated base-change square satisfied the $BC_!$ (resp. BC_* condition).

We associate a norm-diagram to the base-change square

$$\begin{array}{ccc} F_* q_! & \xrightarrow{Nm_q} & F_* q_* \\ \downarrow{\scriptstyle \beta_!} & & \downarrow{\scriptstyle \beta_!} \\ \tilde{q}_! F_* & \xrightarrow[Nm_{\tilde{q}}]{} & \tilde{q}_* F_* \end{array} .$$

Definition ((Weakly) ambidextrous square)

A weakly ambidextrous base-change square is one whose associated norm diagram commutes up to homotopy. A weakly ambidextrous square is ambidextrous if q is iso-normed.

Definition (Integration)

Assume that $q : A \to B$ is iso-normed, for $X, Y \in \mathrm{Fun}(B, \mathcal{C})$ and $f : q^*X \to q^*Y$, we define

$$\int_q : \mathrm{Hom}_{h\,\mathrm{Fun}(A,\mathcal{C})}(q^*X, q^*Y) \to \mathrm{Hom}_{h\,\mathrm{Fun}(B,\mathcal{C})}(X, Y)$$

$$X \xrightarrow{u_*} q_*q^*X \xrightarrow{q_*q^*f} q_*q^*Y \xrightarrow{Nm_q^{-1}} q_!q^*Y \xrightarrow{c_!} Y$$

Let $|q| : 1_{\mathcal{C}} \to 1_{\mathcal{C}}$ denote

$$\int_q q^*\mathrm{Id}_{1_{\mathcal{C}}} = \int_q \mathrm{Id}_{q^*1_{\mathcal{C}}} = c_!^q \circ \mu_q(\mathrm{Id}_{1_{\mathcal{C}}}).$$

In particular, for a space A, let $|A|$ denote $|q|$ for the collapsing map $q : A \to \mathrm{pt}$.

The integration map satisfies a few good properties when q is iso-normed.

Proposition (2.1.14 Homogeneity)

Let $q : A \to B$ be iso-normed and $X, Y, Z \in \mathrm{Fun}(B, \mathcal{C})$.

1. *For all maps $f : q^*X \to q^*Y$ and $g : Y \to Z$, we have*
$$g \circ (\int_q f) = \int_q (q^* g \circ f).$$
2. *For all maps $f : X \to Y$ and $g : q^*Y \to q^*Z$, we have*
$$(\int_q g) \circ f = \int_q (g \circ q^* f).$$

Proposition (Higher Fubini's Theorem)

Let $q : A \to B, p : B \to C$ *both be iso-normed and* $X, Y \in \mathrm{Fun}(B, \mathcal{C}), f : p^* q^* X \to p^* q^* Y$*, we have*

$$\int_q \Big(\int_p f \Big) = \int_{qp} f.$$

We are mainly interested in symmetric monoidal categories.

Definition

(⊗-normed) A map $f : A \to B$ is ⊗-normed if $q^* : \mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C})$ is monoidal, and for all $X \in \mathrm{Fun}(B, \mathcal{C})$, $Y \in \mathrm{Fun}(A, \mathcal{C})$, the compositions

$$q_!(Y \otimes (q^* X)) \to (q_! Y) \otimes (q_! q^* X) \xrightarrow{\mathrm{Id} \otimes c_!} (q_! Y) \otimes X$$

$$q_!((q^* X) \otimes Y) \to (q_! q^* X) \otimes (q_! Y) \xrightarrow{c_! \otimes \mathrm{Id}} X \otimes (q_! Y)$$

are isomorphisms.

Now we inductively define ambidexterity.

Definition (3.1.3 Induced norm)

Let $q : A \to B$ be a map of spaces and let $\delta : A \to A \times_B A$ be the diagnoal of q. Let $\mathcal{C}$ be an ∞-category that admits all q-(co)llimits and δ-(co)limits. Suppose that δ is iso-normed, i. e. there exists an isomorphic natural transformation

$$Nm_\delta : \delta_! \to \delta_*$$

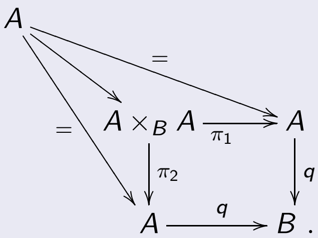
we define the diagonally induced normed map

$$Nm_q : q_! \to q_*$$

as follows:

Definition (cont.)

consider a commutative diagram



Since δ is iso-normed, we have a wrong way unit map

$$\mu_\delta : Id \to \delta_! \delta^*$$

that exhibit $\delta_!$ as the right adjoint to δ^*.
By the previous lemma, the associated base-change square satisfies the $BC_!$ condition, so we can define the composition

Definition (cont.)

$$\nu_q : q^* q_! \xrightarrow{\beta_!^{-1}} (\pi_2)_! \pi_1^* \xrightarrow{\mu_\delta} (\pi_2)_! \delta_! \delta^* \pi_1 \to \mathrm{Id}\,.$$

By applying precomposing q_* on both side, we get

$$Nm_q : q_! \to q_*.$$

Definition ((Weakly) ambidextrous map)

Let $\mathcal{C}$ be an ∞-category for $m \geq -2$ be an integer. A map of spaces $q : A \to B$ is called weakly ambidextrous if

1. $m = -2$, in which case the inverse of q^* is both a left and right adjoint of q^*, in which case the norm map is the identity of the inverse of q^*.
2. $m \geq 1$, and the diagonal $\delta : A \to A \times_B A$ is $(m-1)$-$\mathcal{C}$-ambidextrous, in which case the norm map is the diagonally induced one from the norm of δ.

Proposition

Let $\mathcal{C}$ be a ∞-category and a $\mathcal{C}$-ambidextrous $q : A \to B$ between spaces. For all $X, Y \in \mathrm{Fun}(B, \mathcal{C})$ and $f : q^*X \to q^*Y$, we have

$$s_B^* \int_q f = \int_q s_A^* f.$$

Definition ((F, q)-square)

Given a map $q : A \to B$ between spaces and a functor $F : \mathcal{C} \to \mathcal{D}$ between ∞-categories, we define the (F, q)-square to be the commutative square

$$\begin{array}{ccc} \mathrm{Fun}(B, \mathcal{C}) & \xrightarrow{F_*} & \mathrm{Fun}(B, \mathcal{D}) \\ \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} \\ \mathrm{Fun}(A, \mathcal{C}) & \xrightarrow[F_*]{} & \mathrm{Fun}(A, \mathcal{D}) \end{array}.$$

Proposition ((F, q)-square and ambidexterity)

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories which preserves $(m-1)$-finite colimits. Let $q : A \to B$ be a m-finite map of spaces that are both (weakly) $\mathcal{C}$-ambidextrous and (weakly) $\mathcal{D}$-ambidextrouss, the (F, q)-sqaure satisfies is (weakly) ambidextrous.

Definition (m-semiadditivity)

For $m \geq -2$, a ∞-category $\mathcal{C}$ is m-semiadditive if it admits all m-finite (co)limits and all m-finite maps of spaces are m-ambidextrous. By definition, if $\mathcal{C}$ is m-ambidextrous, it's n-ambidextrous for all $n \leq m$. We call $\mathcal{C}$ ∞-semiadditive if it's m-semiadditive for all m.

Definition (m-semiadditive functor)

Let $\mathcal{C}$ and $\mathcal{D}$ be m-semiadditive ∞-categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called m-semiadditive if it preserves m-finite (co)limits.

Proposition (Naturality in (F, q)-square)

Let $F : \mathcal{C} \to \mathcal{D}$ be an m-semiadditive functor and let $q : A \to B$ be an m-finite map of spaces. For all $X, Y \in \mathrm{Fun}(B, \mathcal{C})$ and $f : q^*X \to q^*Y$, we have

$$F(\int_q f) = \int_q F(f).$$

In particular, $F(|A|_{Id_{\mathcal{C}}}) = |A|_{Id_D}$

Proposition

Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a symmetric monoidal ∞-category. Let $q : A \to B$ be a weakly $\mathcal{C}$-ambidextrous map of spaces such that $\otimes$ distributes over q-colimits. Then q is $\otimes$-normed.

Corollary

Let $F : \mathcal{C} \to \mathcal{D}$ be an m-finite colimit preserving monoidal functor between monoidal categories that admit m-finite colimits and the tensor product distributes over m-finite colimits.

1. *An m-finite map of spaces $q : A \to B$ that is $\mathcal{C}$-ambidextrous and weakly $\mathcal{D}$-ambidextrous is $\mathcal{D}$-ambidextrous.*
2. *If $\mathcal{C}$ is m-semiadditive, so is $\mathcal{D}$.*

Tools: Symmetric monoidal dimension

Definition (m-semiadditively symmetric monoidal ∞-category)

An m-semiadditively symmetric monoidal ∞-category is an m-semiadditive symmetric monoidal ∞-category such that the tensor products distribute over all m-finite colimits.

Definition

(Dimension of a dualizable object) Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ be a symmetric monoidal ∞-category. For a dualization object $X \in \mathcal{C}$, let $X^{\vee}$ denote the dual of X. We denote by $\dim_{\mathcal{C}}(X) \in \mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})$ the composition

$$1_{\mathcal{C}} \xrightarrow{\eta} X \otimes X^{\vee} \xrightarrow{\sigma} X^{\vee} \otimes X \xrightarrow{\epsilon} 1_{\mathcal{C}}$$

where η, ϵ are the coevaluation and evaluation map respectively and σ is the swapping map. We say that a space A is dualizable in $\mathcal{C}$ if 1_A is dualizable $\mathcal{C}$ and we denote

$$\dim_{\mathcal{C}}(A) = \dim_{\mathcal{C}}(1_A) := \dim_{\mathcal{C}}(q_! q^* 1)$$

for $q : A \to \mathrm{pt}$.

Proposition

Let $(\mathcal{C}, \otimes, 1_{\mathcal{C}})$ *be a m-semiadditively symmetric monoidal* ∞*-category. Every m-finite space* A *is dualizable in* $\mathcal{C}$ *and*

$$\dim_{\mathcal{C}}(A) = |A^{S^1}|.$$

In particular, if A *is a loop space, we get*

$$\dim_{\mathcal{C}}(A) = |A||\Omega A|,$$

so we have

$$\dim_{\mathcal{C}}(B^m C_p) = |B^m C_p||B^{m-1} C_p|.$$

Tools: Additive p-derivation and α-operation

Definition (Equavariant power, Θ^p-square)

Given a symmetric monoidal category $\mathcal{C}$, $X, Y \in \mathcal{C}$ and $f : X \to Y$, there is a C_p-equavariant map $f^{\otimes p} : X^{\otimes p} \to Y^{\otimes p}$ where C_p acts on the domain and target by permuting the factors. Therefore, we have a functor

$$\Theta^p : \mathcal{C} \to \mathrm{Fun}(BC_p, \mathcal{C})$$

whose composition with $e^* : \mathrm{Fun}(BC_p, \mathcal{C}) \to \mathcal{C}$ is homotopic to the p-th power functor. For $X \in \mathcal{C}$ (or an object in any symmetric monoidal category), we define

$$X \wr C_p = (X^p)_{hCp}$$

a natural functor (with respect to A and $\mathcal{C}$)

$$\Theta^p_A : \mathrm{Fun}(A, \mathcal{C}) \xrightarrow{(-)\wr C_p} \mathrm{Fun}(A \wr C_p, \mathcal{C} \wr C_p) \xrightarrow{\otimes} \mathrm{Fun}(A \wr C_p, \mathcal{C}).$$

Definition (Θ^p-square of q)

For a map of space $q : A \to B$, by naturality we have a commutative (up to homotopy) square

$$\begin{array}{ccc} \mathrm{Fun}(B, \mathcal{C}) & \xrightarrow{\Theta^p_B} & \mathrm{Fun}(B \wr C_p, \mathcal{C}) \\ \downarrow q^* & & \downarrow q^* \\ \mathrm{Fun}(A, \mathcal{C}) & \xrightarrow[\Theta^p_A]{} & \mathrm{Fun}(A \wr C_p, \mathcal{C}) \end{array} .$$

and we call this the Θ^p-square of q.

Proposition (Θ^p-square BC conditions)

Let $q : A \to B$ be a map of spaces and let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal ∞-category that admits all q-(co)limits. If $\otimes$ distributes over all q-colimits (q-limits), then the Θ^p-square of q satisfies the $BC_!$ (BC_*) condition.

Proposition (Θ^p-square and ambidexterity)

Let $q : A \to B$ be an m-finite map of spaces and let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal ∞-category, then the Θ^p-square is ambidextrous.

Proposition (Naturality in Θ^p-square)

Let $\mathcal{C}$ be a m-semiadditively symmetric monoidal ∞-category and $q : A \to B$ an m-finite map of spaces. For every X, Y in $\mathrm{Fun}(B, \mathcal{C})$ and $f : q^*X \to q^*Y$, we have

$$\Theta^p_B\Big(\int_q f\Big) = \int_{q \wr C_p} \Theta^p_A(f).$$

Definition (Additive p-derivation)

Let R be a commutative ring. An additive p-derivation on R is a function of sets

$$\delta : R \to R$$

such that

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}$$

for all $x, y \in R$ and $\delta(1) = 0$.

Example (Additive p-derivation)

On $\mathbb{Z}_p$, the Fermat quotient

$$\tilde{\delta}(x) = \frac{x - x^p}{p}$$

is a additive p-derivation.

Definition (Rational elements)

Let R be a commutative ring. Let $\phi_0 : \mathbb{Z} \to R$ be the unique ring homomorphism and let S_R be the set of primes p wuch that $\phi_0(r) \in R^\times$. We denote

$$\mathbb{Q}_R = \mathbb{Z}[S_R^{-1}] \subset \mathbb{Q}$$

and let $\phi : \mathbb{Q}_R \to R$ be the unique extension of ϕ_0. We call an element $x \in R$ rational if it is in the image of ϕ.

$$\begin{array}{ccc} \mathbb{Q}_R := \mathbb{Z}[S_R^{-1}] & & \\ \uparrow & \overset{\phi}{\searrow} & \\ \mathbb{Z} & \xrightarrow[\phi_0]{} & R \end{array} .$$

Definition (Detects invertibility)

Given a ring homomorphism $f : R \to S$, we say that f detects invertibility if for every $x \in R$, the invertibility of $f(x)$ implies the invertibility of x.

Theorem (Criterion for invertibility)

Let (R, δ) be a p-local semi-δ-ring and I be the ideal of torsion elements in R. There is a unique additive p-derivation on R/I such that the quotient map $R \to R/I$ is a homomorphism of semi-δ-ring and it detects invertibility.

From now on we assume that $\mathcal{C}$ is a stable 1-semiadditively symmetic monoidal ∞-category with

$$X \in \mathrm{coCAlg}(\mathcal{C}), \;\; Y \in \mathrm{CAlg}(\mathcal{C})$$

and so

$$R = \mathrm{Hom}_{h\mathcal{C}(X,Y)}$$

is a commutative rig (a ring without additive inverse). We denote

$$\mathrm{pt} \xrightarrow{e} BC_p \xrightarrow{r} \mathrm{pt}\,.$$

The $\mathbb{E}_\infty$-coalgebra and $\mathbb{E}_\infty$-algebra structures on X and Y respectively provide symmetric comultiplication and multiplication maps:

$$\overline{t}_X : X \to (X^{\otimes p})^{hCp} = (X^{\otimes p} \times ECp)^{Cp} = r_* \Theta^p(X)$$

$$\overline{m}_Y : r_! \Theta^p(Y) = (Y^{\otimes p})_{hCp} = (Y^{\otimes p} \times ECp)_{Cp} \to Y$$

This maps have mates

$$t_X : r^* X \to \Theta^p(X), \quad m_Y : \Theta^p(Y) \to r^* Y$$

such that

$$e^* t_X : X = e^* r^* X \to e^* \Theta^p(X) = X^{\otimes p}$$

$$e^* m_Y : e^* \Theta^p(Y) \to e^* r^* Y = Y$$

is the ordinary comultiplication and multiplication maps, respectively.

Definition ($\overline{\alpha}, \alpha$ transformation)

Given

$$g : \Theta^p(X) \to \Theta^p(Y)$$

we define $\bar{\alpha}(g) : X \to Y$ to be either of the compositions in the commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\overline{t}_X} & r_*\Theta^p(X) & \xrightarrow{Nm_r^{-1}} & r_!\Theta^p(X) & & \\
 & & \downarrow{\scriptstyle q^*} & & \downarrow{\scriptstyle q^*} & & \\
 & & r_*\Theta^p(Y) & \xrightarrow{Nm_r^{-1}} & r_!\Theta^p(Y) & \xrightarrow{\overline{m}_Y} & Y .
\end{array}$$

Given $f : X \to Y$, we define

$$\alpha(f) = \overline{\alpha}(\Theta^p(f)).$$

Proposition (Functoriality of α)

Let

$$X, X' \in coCAlg(\mathcal{C}),\ \ Y, Y' \in CAlg(\mathcal{C}).$$

Given maps $g : Y \to Y', h : X' \to X$, *we have*

$$\alpha(g \circ f \circ h) = g \circ \alpha(f) \circ h.$$

Proposition (4.2.5 Naturality of α)

Let $F : \mathcal{C} \to \mathcal{D}$ *be a 1-semiadditive symmetric monoidal functor between two 1-semiadditively symmetric monoidal* ∞*-categories. We have a commutative diagram*

$$\begin{array}{ccc} Hom_{h\mathcal{C}}(X, Y) & \xrightarrow{F} & Hom_{h\mathcal{D}}(FX, FY) \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha} \\ Hom_{h\mathcal{C}}(X, Y) & \xrightarrow[F]{} & Hom_{h\mathcal{D}}(FX, FY) \end{array} .$$

Proposition (α is a additive p-derivation)

We have

$$\alpha(f+g) = \alpha(f) + \alpha(g) + \frac{(f+g)^p - f^p - g^p}{p} \in \mathrm{Hom}_{h\mathcal{C}(X,Y)}$$

Proposition (α-action)

We have

$$\alpha(f) = \int_{BC_p} \Theta^p(f)$$

and in particular we get

$$\alpha(|A|) = |A \wr BC_p|.$$

Definition

We define

$$\delta(f) = |BC_p| f - \alpha(f).$$

This is an additive p-derivation on $\mathrm{Hom}_{h\mathcal{C}}(X, Y)$

Tools: Detection principle

Proposition

(More detection principle for higher semiadditivity) Let $m \geq 1$ and let $F : \mathcal{C} \to \mathcal{D}$ be a colimit preserving symmetric monoidal functor between presentably symmetric monoidal, p-local, m-semiadditive ∞-categories. Assume that

1. *the induced map $\varphi : \mathcal{R}_{\mathcal{C}} \to \mathcal{R}_{\mathcal{D}}$ detects invertibility*
2. *the image of $|BC_p|_{\mathcal{D}}, |B^m C_p|_{\mathcal{D}}$ in the ring ${\mathcal{R}_{\mathcal{D}}}^{tf}$ are both rational and non-zero,*

then $\mathcal{C}$ and $\mathcal{D}$ are both $(m+1)$-semiadditive.

Proposition

(Bootstrap Machine)Let $m \geq 1$ and let $F : \mathcal{C} \to \mathcal{D}$ be a colimit preserving symmetric monoidal functor between presentably symmetric monoidal, p-local, m-semiadditive ∞-categories. Assume that

1. *$\mathcal{C}$ is 1-semiadditive.*
2. *The induced map $\varphi : \mathcal{R}_{\mathcal{C}} \to \mathcal{R}_{\mathcal{D}}$ detects invertibility.*
3. *For every $0 \leq k \leq m$, if the space $B^k C_p$ is dualizable in $\mathcal{D}$, then the image of $\dim_{\mathcal{D}}(B^k C_p)$ in ${\mathcal{R}_{\mathcal{D}}}^{tf}$ is dualizable and non-zero.*

Definition (Nil-conservativity)

We call a monoidal colimit-preserving functor $F : \mathcal{C} \to \mathcal{D}$ between two stable presentably monoidal ∞-category nil-conservative if for $F(R) = 0$ implies $R = 0$ for $R \in$Alg$(\mathcal{C})$.

Proposition (Nil-conservativity and invertibility)

Let $F : \mathcal{C} \to \mathcal{D}$ be a nil-conservative functor. The induced ring homomorphism $\mathcal{R}_{\mathcal{C}} \to \mathcal{R}_{\mathcal{D}}$ detects invertibility.

Application to chromatic homotopy theory

Definition

A weak ring is a spectrum R together with a "unit" map and a "multiplication" map

$$u : \mathbb{S} \to R \qquad \mu : R \otimes R \to R$$

such that the composition

$$R \xrightarrow{u \otimes \mathrm{Id}} R \otimes R \xrightarrow{\mu} R$$

is homotopic to the identity.

This is a weaker notion of a ring spectrum, and it follows from the definition that a ring spectrum is a weak ring.

Definition (Chromatic support)

For a $\otimes$-localization functor $L_E : \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{(p)}$, we define the chromatic support

$$\mathrm{supp}(E) := \mathrm{supp}(L_E) = \{0 \neq n \neq \infty : L(K(n)) \neq 0\}.$$

Example (Chromatic support)

1. For a finite spectrum $F(n)$ of type n, by definition $\mathrm{supp}(F(n)) = \{n, n+1, \ldots, \infty\}$.
2. $\mathrm{supp}(K(n)) = \mathrm{supp}(T(n)) = n$.

Proposition

For a $\otimes$-localization functor $L : \mathrm{Sp}_{(p)} \to \mathrm{Sp}_{(p)}$, we have $n \in \mathrm{supp}(L)$ if and only if $\mathrm{Sp}_{K(n)} \subset \mathrm{Sp}_L$.

Corollary

Let R be a weak ring. The functor

$$L : \mathrm{Sp}_R \to \prod_{n \in \mathrm{supp}(R)} \mathrm{Sp}_{K(n)} ,$$

whose n-th component is $K(n)$-localization, is nil-conservative.

Proposition

For every $0 \neq n < \infty$, The functor $\widehat{E}_n : \mathrm{Sp}_{T(n)} \to \widehat{\mathrm{Mod}}_{E_n}$ detects invertibility. Consequently, the canonical map

$$\pi_0 \mathbb{S}_{T(n)} \to \pi_0 E_n$$

it induces detects invertibility.

Further generalization

Theorem (Generalization to weak ring)

Let R be a non-zero p-local weak ring. TFAE:

1. *There exists a unique integer $n \geq 0$, such that*

$$\mathrm{Sp}_{K(n)} \in \mathrm{Sp}_R \in \mathrm{Sp}_{T(n)}.$$

2. *$supp(R) = \{n\}$.*
3. *Sp_R is 1-semiadditive.*
4. *Sp_R is ∞-semiadditive.*