Dieudonné Modules and the Classification of Formal Groups

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Abstract

A Dieudonné module is a module over the ring of Witt vectors over some field k generated by two endomorphisms Frobeius F and Verschiebung V. The category of formal groups over a perfect field k is equivalent to the category of finitely generated Dieudonné module under some mild condition, so the classification of formal groups reduces to the classification of such modules. In this talk, I will introduce the construction of Dieudonné modules and compute some Dieudonné modules associated with various formal group.

1 Motivation

There is a functor

$$DM : \{\text{Formal groups over a perfect group } k\} \to \{\text{Finitely generated free } W_k\text{-modules with } F, V\}$$

This is an equivalence of category, so the classification of formal groups reduces to the classification of W_k-module. I will also introduce the Honda formal group and compute the associated Dieudonné module, which helps us compute the Morava stablizer group.

2 Preliminaries

Definition 1. (Adic R-algebra) For a ring R, then an adic R-algebra A is an augmented R-algebra with augmentation

$$A \xrightarrow{\epsilon} R$$

whose augmentation ideal $ker(\epsilon)$ is nilpotent. We write $\mathcal{J}(A)$ for the augmentation ideal and denote the category of adic R-algebra as Adic_R.

Definition 2. (Formal affine plane) A formal affine plane of dimension n over R is the functor

$$\hat{\mathbb{A}}^n_R : \text{Adic}_R \to \text{pointed sets}$$

$$A \to \text{Hom}^{cts}_R(R[[x_1, x_2, ..., x_n]], A) = \mathcal{J}(A)^n$$

The category FV_R of formal varieties over R is the ctaegory whose objects are functors

$$V : \text{Adic}_R \to \text{pointed sets}$$

isomorphic to $\hat{\mathbb{A}}^n_R$ for some n (the dimension of V) and morphisms are natural transformations. The isomorphism

$$V \xrightarrow{\cong} \hat{\mathbb{A}}^n_R$$

is called a system of parameters. The isomorphism

$$\hat{\mathbb{A}}^n_R \xrightarrow{\cong} V$$

is called a coordinate.

Definition 3. (Formal group, formal group law) A formal group over R is a group object G in the category of formal varieties over R. A formal group law over R is a formal group together with a system of parameters, i.e. an isomorphism

$$G \xrightarrow{\cong} \hat{\mathbb{A}}^n_R$$

of set-valued functors.

Now we introduce two basic examples of formal groups.

Example 4. The additive formal group $\hat{\mathbb{G}}_a$ is the functor

$$\hat{\mathbb{G}}_a(A) = \mathcal{J}(A)$$

and a coordinate is the identity map

$$\mathcal{J}(A) \to \mathcal{J}(A).$$

The multiplicative formal group $\hat{\mathbb{G}}_m$ is the functor

$$\hat{\mathbb{G}}_m(A) = (1 + \mathcal{J}(A))^{\times}$$

and a possible coordinate is

$$a \to 1 - a.$$

3 Witticism

Definition 5. (Witt scheme) Let $\mathbb{A}^n$ be the scheme

$$\mathbb{A}^n := \text{Spec } \mathbb{Z}[x_0, x_1, ..., x_{n-1}].$$

The Witt scheme $\mathbb{W}_{p,n}$ is a ring scheme whose underlying scheme is $\mathbb{A}^n$, with a morphism

$$\mathbb{W}_{p,n} \xrightarrow{w} \mathbb{A}^n$$

which is given on the coordinates by the Witt polynomial

$$\begin{aligned} w_0(x_0, ...x_{n-1}) &= x_0 \\ w_1(x_0, ...x_{n-1}) &= x_0^p + px_1 \\ w_2(x_0, ...x_{n-1}) &= x_0^{p^2} + px_1^p + p^2x_2 \\ w_j(x_0, ...x_{n-1}) &= \sum_{d=0}^{j} p^d x_d^{p^{j-d}} \end{aligned}$$

We take

$$\mathbb{W}_p(A) = \varprojlim \mathbb{W}_{p,n}(A).$$

We can generalize this notion to an arbitrary field k and get a $\mathbb{W}_p k$ by the following pullback diagram

$$\begin{array}{ccc} \mathbb{W}_p k & \longrightarrow & \mathbb{W}_p \\ \downarrow & \lrcorner & \downarrow \\ \text{Spec}\, k & \longrightarrow & \text{Spec}\, \mathbb{Z} \end{array}$$

When $k = \mathbb{F}_p^n$, We can think of $\mathbb{W}_p k$ with respect to k as the p-adic interger with respect to $\mathbb{F}_p$

Definition 6. (Frobenius, Verschiebung) For an R-valued point $(a_0, a_1, ...)$ of $\mathbb{W}_p$ (= homomorphism from $\mathbb{Z}[x_0, ...]$ to R), then the components of its image under w

$$(w_0(a), w_1(a), ...)$$

are called the ghost components, and the ghost components completely characterize the point. To be defined in terms of the original Witt components,

$$V(x_0, x_1, ...) = (0, x_0, x_1, ...)$$

ghost components, the operator V on $\mathbb{W}_p$ is given by

$$V^w(0, w_0(x_0, x_1, ...), w_1(x_0, x_1, ...), ...) = (0, pw_0, pw_1, ...).$$

It's difficult to write down what the operator F does on the Witt component, but with respect to the ghost components, we get

$$F^w(w_0, w_1, w_2...) = (w_1, w_2...)$$

Definition 7. (Witt formal group) We can make the Witt scheme $\mathbb{W}_{n,p}$ into a formal group isomorphic as a formal variety to $\hat{\mathbb{A}}^n_R$ by completion at the origin. The p-typical Witt formal group is defined

$$\hat{\mathbb{W}}_p(A) = \varprojlim \hat{\mathbb{W}}_{p,n}(A)$$

Thus, a point in $\hat{\mathbb{W}}_p(A)$ looks like $(a_1, ..., a_n)$ with $a_i \in \mathcal{J}(A)$ and there exists $N \in \mathbb{N}$ such that $a_i = 0$ for $i > N$. There is also the global Witt formal group $\hat{\mathbb{W}}(A)$ that is defined similarly, and later we will see that it plays an important role in the curve group. Analogous to the p-typical Witt vectors, We have operator V_r, F_r (with grading r) that act on the global Witt vectors. This is rather simple with respect to the ghost component:

$$V_r^w(w_1, w_2...) = (0, 0, ..., 0, w_1, w_2...)$$
$$F_r^w(w_1, w_2...) = (w_r, w_{2r}...).$$

4 Construction

Definition 8. (Dieudonné module) Suppose that k is a perfect field of characteristic p and $\varphi : a \to a^p$ is the Frobenius isomorphism. A Dieudonné module over k is a $\mathbb{W}_p k$-module M equipped with operators F and V which are $\mathbb{W}_p k$-linear maps

$$F : \varphi^* M \to M$$
$$V : M \to \varphi^* M$$

which satisfy

$$FV = p = VF.$$

Moreover, M is required to be uniform and reduced with respect to V, i.e.

$$M \cong \varprojlim M/V^r M, M/VM \cong V^k M/V^{k+1} M \ (k > 0).$$

Theorem 9. *Suppose that k is a perfect field of characteristic $p > 0$. There is a functor DM inducing an equivalence of categories*

$$DM : \mathrm{FG}_k \xrightarrow{\cong} \textit{Dieudonné modules over } k$$

We introduce a few tools before we can state what the functor DM does to a formal group.

Definition 10. (Curve functor, F,V on curve) For a formal group G, the group of curves in G is the group

$$C(G) = C_R(G) := \mathrm{Hom}_{\mathrm{FV}}(\hat{\mathbb{A}}^1_R, G).$$

There are three basic types of endomorphisms of curves: homotheties, Verschiebung, and Frobenius. Given a curve

$$\gamma : \hat{\mathbb{A}}^1_R \to G$$

we have new curves by

$$[a](\gamma) : \hat{\mathbb{A}}^1_R \xrightarrow{a} \hat{\mathbb{A}}^1_R \xrightarrow{\gamma} G$$
$$V_r(\gamma) : \hat{\mathbb{A}}^1_R \xrightarrow{t \to t^r} \hat{\mathbb{A}}^1_R \xrightarrow{\gamma} G.$$

The action of Frobenius is more complicated, we have

$$F_n\gamma(t) = \prod_{i=0}^{n-1}(1-\zeta^i t^{1/n})$$

where ζ is the n-th root of unity. Recall that F_r, V_r also acts on the global Witt formal group, and we will see the relationship shortly.

Definition 11. (p-typical curve) A curve γ is called p-typical if $F_n\gamma = 0$ when n is not a power of p.

Theorem 12. *(Global Witt formal group represents curves) Let $\hat{\mathbb{W}}$ be the Witt formal group, and γ_1 be the curve*

$$\gamma_1 : \hat{\mathbb{A}}^1 \to \hat{\mathbb{W}}$$

given by $t \to (t, 0, 0, ...)$. For a formal group G over the ring R, precomposing with γ_1 induces an isomorphism

$$\mathrm{Hom}_{\mathrm{FG}_R}[\hat{\mathbb{W}}, G] \cong CG.$$

Under this isomorphism, the operator F_r on curves corresponds to the operator V_r on the Witt formal group. When $r = 0$ in R, then the operator V_r on curves corresponds to F_r on the Witt formal group.

We have $DM(G) = \{\gamma : \gamma \in CG : F_n\gamma = 0, n \neq p^m\} = \mathrm{Hom}_{\mathrm{FG}_R}[\hat{\mathbb{W}}_p, G] \subset \mathrm{Hom}_{\mathrm{FG}_R}[\hat{\mathbb{W}}, G] = CG$.

5 Examples

Example 13. (Additive formal group) For an adic algebra A, the multiplicative group is

$$\hat{\mathbb{G}}_a(A) = \mathcal{J}(A)$$

with parameter

$$\hat{\mathbb{A}}^1 \xrightarrow{\gamma} \hat{\mathbb{G}}_a$$

$$a \to a.$$

We have

$$F_n\gamma(t) = \sum_{j=0}^{n-1} t^{1/n}\zeta^j = 0$$

through direct computation, so the Frobenius F_n vanishes on γ for any $n > 1$ and $V\gamma(t) = t^n$. For a field k with characteristic p, the Dieudonné module of the additive group is

$$\prod_{n\geq 0} V^n k$$

where the Frobenius acts trivially, and the Witt vectors act through the projection $\mathbb{W}_p k \to k$.

Example 14. (Multiplicative formal group) Recall that for an adic algebra A, the multiplicative group is

$$\hat{\mathbb{G}}_m(A) = (1 + \mathcal{J}(A))^\times$$

with parameter

$$\hat{\mathbb{A}}^1 \xrightarrow{\gamma} \hat{\mathbb{G}}_m$$

$$a \to 1 - a.$$

We get

$$\gamma(r) +_{\hat{\mathbb{G}}_m} \gamma(s) = (1-s)(1-t) = \gamma(s+t-st)$$

and

$$F_n\gamma(t) = \prod_{i=0}^{n-1}(1-\zeta^i t^{1/n}) = \gamma(t)$$

through direct computation. We can produce a p-typical curve $\tilde{\gamma}$ and show that $F\tilde{\gamma} = \tilde{\gamma}$. Since

$$V\tilde{\gamma} = VF\tilde{\gamma} = p\tilde{\gamma}.$$

Therefore, we know that for a $\mathbb{Z}_{(p)}$-algebra R, the Dieudonné module for $\hat{\mathbb{G}}_m$ is free of rank 1 over $\mathbb{W}_p R$, generated by the curve $\tilde{\gamma}$ with F and V acting by

$$F\tilde{\gamma} = \tilde{\gamma}, V\tilde{\gamma} = p\tilde{\gamma}.$$

Example 15. (Formal group of each height over a perfect field k) We will describe a Dieudonné module of height h for each finite positive height h corresponding to a one-dimensional formal group of height h. We are able to find a good basis for a Dieudonné module of heigh h:

$$\gamma, V\gamma, ..., V^{h-1}\gamma$$

where γ is a p-typical curve. For $i \geq 1$, we have $FV^i\gamma = pV^{h-1}\gamma$, so the remaining indeterminacy is $F\gamma$. We can choose $F\gamma = V^{h-1}\gamma$ that satisfies the relations of F, V, so we get

$$V^h\gamma = VV^{j-1}\gamma = VF\gamma = p\gamma$$

so we can write

$$V = \begin{pmatrix} & p \\ I & \end{pmatrix}, F = \begin{pmatrix} & pI \\ 1 & \end{pmatrix}$$

Note that when k is algebraically closed, we can prove that this is the unique formal group law of height h over k up to isomorphism. This is consistent with what we already know about formal group law.

Example 16. (Endomorphism) We start with the formal group Γ with basis

$$\gamma, V\gamma, ..., V^{h-1}\gamma$$

and

$$F\gamma = V^{h-1}\gamma, V^h\gamma = p\gamma.$$

Let k be a finite field and let Γ' be an arbitrary formal group over k. We know there is an isomorphism

$$\phi : \Gamma \cong \Gamma'$$

over $\bar{k}$. Suppose that $g \in \mathrm{Gal}(\bar{k}/k)$, we know g fixes k, so

$$\Gamma = g^*\Gamma, \Gamma' = g^*\Gamma'$$

We want ϕ to be an isomorphism over k, which means k has to be invariant under g, so $\phi = g^*\phi$. For any g, we can associate

$$A_g := g^*\phi \ \circ \phi^{-1} \in \mathrm{Aut}\,\Gamma$$

and we can check that this construction is a homomorphism from $\mathrm{Gal}(\bar{k}/k)$ to $\mathrm{Aut}\,\Gamma$. There is an algebraic result that states that $\Gamma \cong \Gamma'$ over k if and only if $A_g = g^*m \circ m^{-1}$, so we can classify formal groups of height h over k by computing. $H^1(\mathrm{Gal}(\hat{k}/k), \mathrm{Aut}\,\Gamma)$. Therefore, by computing the automorphism group of the Dieudonné modules of Γ and how $\mathrm{Gal}(\bar{k}/k)$ acts on it, we can classify formal groups over heigh h over a finite field k. Moreover, this helps us compute the Morava stablizer group.