Galois Extensions in Stable Homotopy Theory

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UIUC Graduate Student Homotopy Theory Seminar, Spring 2024

April 6, 2024

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Definitions

Let $f : R \to T$ be a map of commutative rings which makes T a commutative R-algebra. Let G a finite group acting on T from the left through the R-algebra homomorphism. Let

$$i : R \to T^G$$

be the inclusion of fixed ring, and let

$$h : T \otimes_R T \to \prod_G T$$

be the commutative ring homomorphism that assigns

$$t_1 \otimes t_2 \mapsto \{g \mapsto t_1 \cdot g(t_2)\}$$

Definition (Galois extension of commutative rings)

We say $f : R \to T$ is a G-Galois extension if the two maps i, h are isomorphisms.

Definitions

Definition (Galois extension of commutative ring spectra)

Let G be a finite group. A map $A \to B$ of commutative ring spectra is a G-Galois extension if the following conditions hold:

1. G acts on B through A-algebra maps.
2. The natural map $i : A \to B^{hG}$ is a weak equivalence.
3. There is a weak equivalence $h : B \wedge_A B \cong \prod_G B$.

The Galois extension is E-local if i, h are E-equivalences.

Definition (Faithful Galois extension)

We say a Galois extension $f : A \to B$ is faithful if B is a faithful A-module. In other words, for any A-module M, we have $B \wedge_A M \cong *$ if and only if $B \cong *$.

Definition

Proposition

Let G be a finite group and B be a normal subgroup of G. If $A \to B$ is a G-Galois extension, then $A \to B^{hN}$ is a G/N-Galois extension.

Examples: Eilenberg–MacLane spectra

Proposition

Let $f : R \to T$ be map of commutative rings with a finite group G acting on T through R-algebra homomorphisms. Then $f : R \to T$ is a G-Galois extension of commutative rings if and only if the induced map $fH : HR \to HT$ is a G-Galois extension of commutative ring spectra.

Proof. Use homotopy fixed point spectral sequence

$$E_2^{p,q} = H^p(G, \pi_q(HT)) \Rightarrow \pi_{q-p}(HT^{h\mathbb{Z}/2})$$

Examples: Topological K-Theory

Proposition

The complexification map $c : KO \to KU$ is a faithful $\mathbb{Z}/2$-Galois extension.

Proof. Use homotopy fixed point spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}/2, \pi_q(KU)) \Rightarrow \pi_{q-p}(KU^{h\mathbb{Z}/2})$$

Pro-Galois Extension

Definition (Pro-finite Galois extension)

Let A be a E-local ring spectrum and a directed system of E-local G_α-Galois extension $A \to B_\alpha$. We also suppose that $A \to B_\alpha$ is a sub-Galois extension of $A \to B_\beta$ is a sub-Galois extension, i. e. there is a chosen surjection $G_\beta \to G_\alpha$ with kernel $K_{\alpha\beta}$ such that $B_\alpha \to B_\beta^{hK_{\alpha\beta}}$ is a weak equivalence. Let $G = \lim_\alpha G_\alpha$ and $B = \operatorname{colim}_\alpha B_\alpha$, then we call $A \to B$ a pro-G-Galois extension.

$$\begin{tikzcd}
 & & A \arrow[dll] \arrow[dl] & \\
B_\alpha \arrow[r] & B_\beta \arrow[r] & \cdots \arrow[r] & \operatorname{colim} B_\alpha =: B
\end{tikzcd}$$

$$G_\alpha \longleftarrow G_\beta \longleftarrow \cdots \longleftarrow \lim G_\alpha =: G$$

In classical Galois theory for fields, each closed subgroup (under the Krull topology) of the Galois group $\mathrm{Gal}(F/E)$ of field extension $E \subset F$ corresponds bijectively to the intermediate field that is fixed by the group, and the finite extension corresponds to the open subgroups of $\mathrm{Gal}(F/E)$. Furthermore, $\mathrm{Gal}(F/E)$ acts continuously on $\bar{F}$ with the discrete topology, so $\bar{F}$ is the union of $\bar{F}^U$ over all the open subgroups U.

Examples: $L_{K(n)}\mathbb{S} = E_n^{h\mathbb{G}_n}$

For a closed subgroup $K \subset \mathbb{G}_n$ and an appropriate filtration of normal open subgroup $\{U_i\} \subset K$, Devinatz and Hopkons define

$$E_n^{dhK} := L_{K(n)}(\mathrm{colim}_i\, U_n^{hU_iK}).$$

They showed that the definition is independent from the choice of $\{U_i\}$ and that it behaves like a continuous homotopy fixed point:

1. For a finite subgroup $K \subset \mathbb{G}_n$, this construction of E_n^{dhK} agrees with $F(EK_+, E_n)^K$.
2. There is a spectral sequence

$$E_2^{p,q} = H^p_{cts}(K, \pi_q(E_n)) \Rightarrow \pi_{q-p}(E_n^{dhK})$$

3. E_n^{dhK} has a residual action by $W(K)$.

Theorem (5.4.4 Devinatz–Hopkins)

1. *For each pair of closed subgroups $H \subset K \subset \mathbb{G}_n$ with H a normal subgroup of finite index in K, the map $E_n^{hK} \to E_n^{hH}$ is a $K(n)$-local K/H-Galois extension. In particular, for each finite subgroup $K \in \mathbb{G}_n$, the map $E_n^{hK} \to E_n$ is a $K(n)$-local K-Galois extension.*
2. *Likewise, for each open normal subgroup $U \subset \mathbb{G}_n$, the map*
$$L_{K(n)}S = E_n^{h\mathbb{G}_n} \to E_n^U$$
is a $K(n)$-local $\mathbb{G}_n/U$-Galois extension.
3. *A choice of descending sequence $\{U_i\}$ of open normal subgroups of $\mathbb{G}_n$ with $\bigcap_i U_i = \{e\}$ exhibits*
$$L_{K(n)}S = E_n^{h\mathbb{G}_n} \to E_n$$
as a $K(n)$-local pro-$\mathbb{G}_n$-Galois extension.

Examples: $K(1)$-local case

The height 1 Honda formal group law over $\mathbb{F}_{p^n}$ is the multiplicative formal group law, and its universal deformation is the multiplicative formal group law over $\mathbb{Z}_p$. The Lubin–Tate spectrum E_1 equals the p-completed complex tological K-theory KU_p, and the Morava stabilzer group $\mathbb{G}_1 = \mathbb{S}_1$ is the group of p-adic units $\mathbb{Z}_p^\times$. An element $k \in \mathbb{Z}_p^\times$ acts on KU_p by the p-adic Adams operation

$$\psi^k : KU_p \to KU_p.$$

The homotopy fixed point spectrum

$$E_1^{h\mathbb{G}_n} = (KU_p)^{h\mathbb{Z}_p^\times}$$

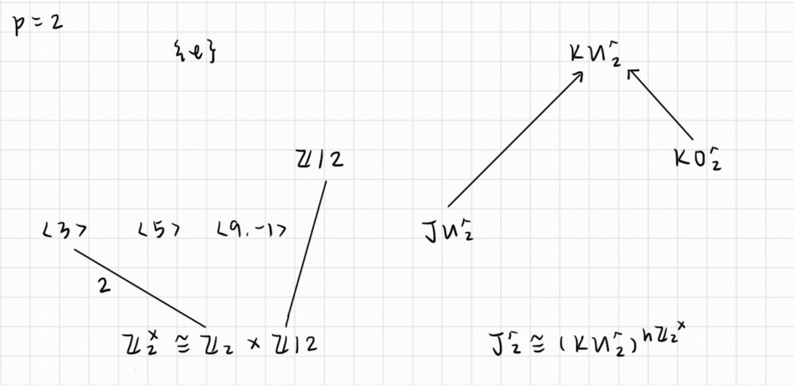
is the p-complete image-of-J spectrum, defined as the fiber of

$$J_2 \to KO_2 \xrightarrow{\psi^3 - 1} KO_2$$

$$J_p \to KU_p \xrightarrow{\psi^r - 1} KU_p$$

for odd p and a topological generator r of $\mathbb{Z}_p^\times$.

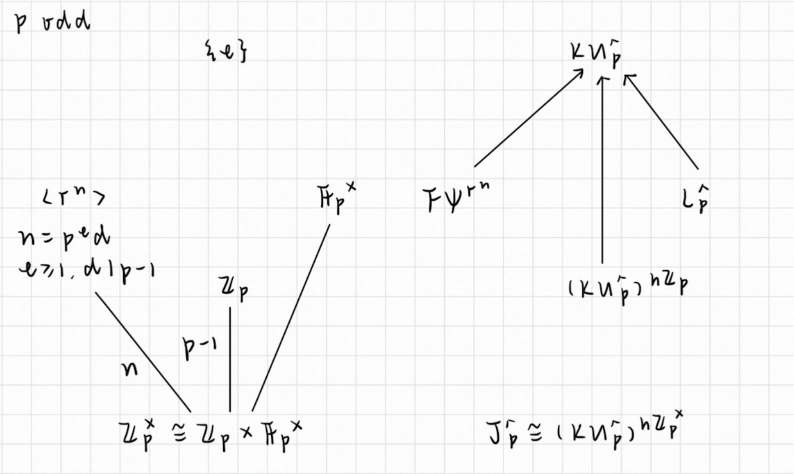
Examples: $K(1)$-local case



$\mathbb{Z}/2-$*Galois extension?*

$$\begin{tikzcd} J_2 \arrow[r] \arrow[d] & KO_2 \arrow[r, "\Psi^3-1"] \arrow[d] & KO_2 \arrow[d] \\ JU_2 \arrow[r] & KU_2 \arrow[r, "\Psi^3-1"] & KU_2 \end{tikzcd}$$

Examples: $K(1)$-local case



$$F\Psi^{r^n} \to KU_p \xrightarrow{\Psi^{r^n}-1} KU_p$$

Cyclotomic extension

Recall that we have an adjunction

$$\mathbb{1}[-] : \mathsf{Sp}^{cn} \rightleftharpoons \mathrm{CAlg}(\mathcal{C}) : (-)^{\times}$$

Definition (Roots of unity)

For a additive, presentably symmetric monoidal category $\mathcal{C}$ and $R \in \mathsf{CAlg}(\mathcal{C})$, for $m \in \mathbb{N}$, we define the space of m-th roots of unity in R by

$$\mu_m(R) = \mathrm{Hom}_{\mathsf{Sp}^{cn}}(HC_m, R^{\times})$$

where C_m is the cyclic group of order m. By the adjunction above, this is correpresented by $\mathbb{1}[C_m]$, i. e.

$$\mu_m(R) = \mathrm{Hom}_{\mathrm{CAlg}(\mathcal{C})}(\mathbb{1}[C_m], R).$$

Cyclotomic extension

We can generalize this construction when $\mathcal{C}$ is higher semiadditive.

Definition ((Higher) cyclotomic extension)

For a stable n-semiadditively symmetric monoidal category $\mathcal{C}$, a prime p and $m \in \mathbb{N}$, we define the space of p-th roots of unity of heigh n in R by

$$\mu^n_{p^r}(R) = \mathrm{Hom}_{\mathrm{Sp}^{cn}}(HC_{p^r}, \Omega^n R^\times).$$

Similarly, this is correpresented by $\mathbb{1}[B_m C_{p^r}]$, i.e.

$$\mu^n_{p^r}(R) = \mathrm{Hom}_{\mathrm{CAlg}(\mathcal{C})}(\mathbb{1}[C_{p^r}], \Omega^n R) = \mathrm{Hom}_{\mathrm{CAlg}(\mathcal{C})}(\mathbb{1}[B^n C_{p^r}], R).$$

Cyclotomic extension

We also have a notion of primitive roots of unity.

Definition (Primitive roots of unity)

We say a higher root of unity $C_{p^r} \to \Omega^n R^\times$ is primitive, if R is of height n and there is no non-0 commutative R-algebra S that satisfy the commutative diagram

$$\begin{tikzcd} C_{p^r} \arrow[r] \arrow[d] & \Omega^n R^\times \arrow[d] \\ C_{p^{r-1}} \arrow[r] & \Omega^n S^\times \end{tikzcd}$$

Cyclotomic extension

For $k \leq r$, we have an injective group homomorphism $C_{p^{r-k}} \hookrightarrow C_{p^r}$. By precomposition, we have

$$\mu^n_{p^r}(R) = \mathrm{Hom}_{\mathrm{Sp}^{cn}}(HC_{p^r}, \Omega^n R^\times) \to \mathrm{Hom}_{\mathrm{Sp}^{cn}}(HC_{p^{r-k}}, \Omega^n R^\times) = \mu^n_{p^{r-k}}(R).$$

We can think of this as raising a p^r-th roots of unity to the p^k-th power to get a p^{r-k}-th roots of unity.
For $k \leq r$, we have a surjective group homomorphism $C_{p^r} \twoheadrightarrow C_{p^k}$. By precomposition, we have

$$\mu^n_{p^k}(R) = \mathrm{Hom}_{\mathrm{Sp}^{cn}}(HC_{p^k}, \Omega^n R^\times) \to \mathrm{Hom}_{\mathrm{Sp}^{cn}}(HC_{p^r}, \Omega^n R^\times) = \mu^n_{p^r}(R).$$

We can think of this as the inclusion of p^k-th roots of unity into p^r-th roots of unity. The group homomorphism $q_n : C_{p^r} \twoheadrightarrow C_{p^k}$ induces a map

$$\overline{q_n} : \mathbb{1}[B^n C_{p^r}] \twoheadrightarrow \mathbb{1}[B^n C_{p^k}].$$

Cyclotomic extension

In particular, let's take $k = r - 1$. The map

$$\overline{q_n} : \mathbb{1}[B^n C_{p^r}] \twoheadrightarrow \mathbb{1}[B^n C_{p^{r-1}}]$$

correpresents the inclusion of roots of unity. When $\mathcal{C}$ is n-semiadditive with semiadditively height n, there exists an idempotent element $\epsilon \in \pi_0(\mathbb{1}[B^n C_{p^r}])$ such that

$$\mathbb{1}[B^n C_{p^r}][\epsilon^{-1}] = \mathbb{1}[B^n C_{p^{r-1}}]$$

and the localization map is identified with $\overline{q_n}$.

Cyclotomic Extension

Example (Chromatic cyclotomic extension)

For $\mathcal{C}$ is n-semiadditive with semiadditively height n, we define

$$\mathbb{1}[\omega^n_{p^r}] := \mathbb{1}[B^n C_{p^r}][(1-\epsilon)^{-1}], \quad R[\omega^n_{p^r}] := R \otimes \mathbb{1}[\omega^n_{p^r}]$$

for $R \in \mathrm{CAlg}(\mathcal{C})$. We call this the height n p^r-th cyclotomic extension of R, and it represents the primitive p^r-th roots of unity.

Proposition

It is a faithful Galois extension in $\mathrm{Sp}_{K(n)}$ and $\mathrm{Sp}_{K(n)}$ with Galois group $\mathbb{Z}/p^{r\times}$.

Application: Cyclotomic Completeness

We have a canonical map

$$\mathbb{1}\left[\omega^{n}_{p^{r-1}}\right] \to \mathbb{1}\left[\omega^{n}_{p^{r}}\right]$$

that corepresents the natural transformation $\omega \to \omega^{p}$ on primitive roots of unity. Now we take the direct limit of this system.

Definition (Infinite cyclotomic extension)

For $\mathcal{C}$ is n-semiadditive with semiadditively height n, we define

$$\mathbb{1}\left[\omega^{\infty}_{p^{r}}\right] := \lim_{r \in \mathbb{N}} \mathbb{1}\left[\omega^{n}_{p^{r}}\right]$$

By definition, this is a pro-$\mathbb{Z}/p^{r\times}$-Galois extension. It's natural to question whether this is (possibly) a Galois extension, i.e. whether

$$\mathbb{1} \to \mathbb{1}\left[\omega^{\infty}_{p^{r}}\right]^{h\mathbb{Z}_{p}^{\times}}$$

is an equivalence. If this is true, we call $\mathcal{C}$ cyclotomic complete.

Application: Cyclotomic Completeness

Example ($\mathrm{Sp}_{K(n)}$ is cyclotomically complete)

Let $\mathcal{C} = \mathrm{Sp}_{K(n)}$, and we have a pro-$\mathbb{Z}_p^\times$ extension

$$L_{K(n)}S^0 \to L_{K(n)}S^0[\omega_{p^r}^\infty].$$

By Devinatz and Hopkins, we have

$$L_{K(n)}S^0 = E_n^{h\mathbb{G}_n}$$

and $\mathbb{Z}_p^\times \subset \mathbb{G}_n$, so for the intermediate extension we have

$$L_{K(n)}S^0 \cong L_{K(n)}S^0[\omega_{p^r}^\infty]^{h\mathbb{Z}_p{}^\times}.$$